



Theorem: If H is a normal subgroup of a solvable group G , then G/H is also solvable.

Proof: Let $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_n = \{e\}$ be a solvable series for G . We consider the series:

$$G/H = G_0/H \supseteq G_1/H \supseteq G_2/H \supseteq \dots \supseteq G_n/H = \{H\}$$

For each $i = 0, 1, 2, \dots, n-1$, G_{i+1} is normal in G_i . Let $x \in G_i/H$. Then $x = gh$ where $g \in G_i$, $h \in H$. Thus

$$\begin{aligned} x G_{i+1}/H &= gh G_{i+1}/H \\ &= gh H G_{i+1}/H \quad (\because H \text{ is normal}) \\ &= g H G_{i+1}/H \\ &= g G_{i+1}/H \\ &= G_{i+1} g H/H \quad (\because G_{i+1} \trianglelefteq G_i, g \in G_i) \\ &= G_{i+1} g h H/H \\ &= G_{i+1} H g h/H \\ &= G_{i+1} H x/H \end{aligned}$$

Therefore $G_{i+1}H \trianglelefteq G_iH$

$$\text{Hence } G_iH/G_{i+1}H = \frac{G_iH/H}{G_{i+1}H/H}$$

$$\text{We define } f: G_i \rightarrow \frac{G_iH}{G_{i+1}H}$$

$$\text{by } f(x) = G_{i+1}Hx \quad \forall x \in G_i$$

$$\begin{aligned} \text{Now } f(xy) &= G_{i+1}Hxy = (G_{i+1}Hx)(G_{i+1}Hy) \\ &= f(x)f(y) \end{aligned}$$

"You must expect great things of yourself before you can do them." - Michael Jordan



Date ! !

Mon ☐ Tue ☐ Wed ☐ Thu ☐ Fri ☐ Sat ☐ Sun ☐

$\therefore f$ is a homomorphism.

Let $y \in G_i H = H G_i$

Then $y = hg$; $h \in H, g \in G_i$

Now $G_{i+1} H = G_{i+1} H h g$

$$= G_{i+1} H g$$

$$= f(g)$$

So f is onto.

Let $g \in G_{i+1}$

Then $f(g) = G_{i+1} H g$

$$= G_{i+1} g H$$

$$= G_{i+1} H, \text{ identity of } \frac{G_i H}{G_{i+1} H}$$

$\therefore g \in \ker f$

So $G_{i+1} \subseteq \ker f$

Hence f induces a homomorphism

$$\bar{f}: G_i / G_{i+1} \rightarrow G_i H / G_{i+1} H$$

such that $\bar{f}(G_{i+1} x) = G_{i+1} H x$

This $\bar{f}$ is onto. Thus $G_i H / G_{i+1} H$ is a homomorphic image of the abelian group G_i / G_{i+1} . So $G_i H / G_{i+1} H$ is abelian.

$$\text{Hence } \frac{G}{H} = \frac{G_0}{H} \supseteq \frac{G_1 H}{H} \supseteq \frac{G_2 H}{H} \supseteq \dots \supseteq \frac{G_n H}{H} = \frac{G}{H} = \{e\}$$

is a solvable series for G/H

consequently G/H is solvable.

"What seems to us as bitter trials are often blessings in disguise." - Oscar Wilde



Theorem

Let H be a normal subgroup of a group G . If both H and G/H are solvable, then G is solvable.

Proof: Let $G/H = G_0/H \supseteq G_1/H \supseteq \dots \supseteq G_k/H = \{\bar{e}\}$ be a solvable series for G/H . Since each G_{i+1}/H is normal in G_i/H , G_{i+1} is normal in G_i . Again $G_k/H = \{\bar{e}\} = \{H\}$ implies $G_k = H$.

Now let $H = H_0 \supseteq H_1 \supseteq \dots \supseteq H_s = \{e\}$ be a solvable series for H . Then $G = G_0 \supseteq G_1 \supseteq \dots \supseteq G_k = H \supseteq H_1 \supseteq \dots \supseteq H_s = \{e\}$ is a solvable series for G . Hence G is a solvable ~~series~~ group.

Corollary

Let H and K be two groups and $G = H \times K$. Then G is solvable iff H and K are solvable.

Proof: Let G be a solvable group. Subgroup of a solvable group is solvable, so $H \times \{e\}$ is solvable. As $H \cong H \times \{e\}$, H is solvable. Similarly K is solvable.

Conversely let H and K be solvable. Let $H' = H \times \{e\} \cong H$. H' , being



Date ! !

isomorphic to H , is solvable. Moreover $G/H' \cong K$ gives G/H' is solvable. Hence G is solvable.

Corollary:

Any p -group is solvable.

Proof: Let G be a p -group. Let $O(G) = p^n$ where p is a prime and $n \geq 1$. If $n=1$, then G is abelian and hence solvable.

Let $n > 1$, we shall apply induction on n . We assume that the statement is true for groups of order p^m with $m < n$.

Now the centre of G , $Z(G)$ is not equal to $\{e\}$. So $O(Z(G)) = p^k$ for some $k \geq 1$. Then $O(G/Z(G)) = \frac{O(G)}{O(Z(G))} = \frac{p^n}{p^k} = p^{n-k}$ where $n-k < n$.

So by induction hypothesis $G/Z(G)$ is solvable. Again $Z(G)$ being abelian, is solvable. Hence G is solvable.

n -th commutator subgroup:

$$G^0 = G$$

$G^{(n+1)} = (G^{(n)})'$ is the commutator subgroup of $G^{(n)}$. $G^{(n)}$ is called the n -th commutator subgroup of G .

"What seems to us as bitter trials are often blessings in disguise." - Oscar Wilde



Theorem:

A group G is solvable if and only if $G^{(n)} = \{e\}$, for some non negative integer n .

Proof: Let G be a solvable group and let $G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_n = \{e\}$ be a solvable series.

We prove inductively that $G^{(k)} \subseteq G_k \forall k$.
Clearly $G^{(0)} = G = G_0$

We assume that $G^{(k)} \subseteq G_k \forall k$.

Now $G^{(k+1)} = (G^{(k)})' \subseteq G_k'$

As G_k/G_{k+1} is abelian, $G_k' \subseteq G_{k+1}$.

Consequently $G^{(k+1)} \subseteq G_{k+1}$

Hence by induction, $G^{(k)} \subseteq G_k \forall k = 0, 1, 2, \dots, n$.

Hence $G^{(n)} \subseteq G_n = \{e\}$

$\therefore G^{(n)} = \{e\}$.

Conversely let $G^{(n)} = \{e\}$, for

some n . Then

$G = G_0 \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \dots \supseteq G^{(n)} = \{e\}$

is a subnormal series for G such

that $G^{(i)}/G^{(i+1)} = G^{(i)}/(G^{(i)})'$ is abelian.

(Since for any subgroup H/H' is abelian). Hence G is solvable.